Lecture 6
Quine-McCluskey Method

- A systematic simplification procedure to reduce a minterm expansion to a minimum sum of products.
- Use $XY + XY' = X$ to eliminate as many as literals as possible.
  - The resulting terms = prime implicants.
- Use a prime implicant chart to select a minimum set of prime implicants.

Determination of Prime Implicants

√ Eliminate literals

Two terms can be combined if they differ in exactly one variable.

AB'CD' + AB'CD = AB'C

$\underbrace{\underline{1\ 0\ 1}}_{X}\ \underbrace{0}_{Y} + \underbrace{\underline{1\ 0\ 1}}_{X}\ \underbrace{1}_{Y'} = \underbrace{\underline{1\ 0\ 1}}_{X}$

A'BC'D + A'BCD' (won't combine)

0 1 0 1 + 0 1 1 0 (check # of 1's)

We need to compare and combine whenever possible.

Sorting to Reduce Comparisons

- √ Sort into groups according to the number of 1's.

$F(a,b,c.d) = \Sigma m(0,1,2,5,6,7,8,9,10,14)$

- No need for comparisons
  - (1) Terms in nonadjacent group
  - (2) Terms in the same group

Group	0	0	0000
Group	1	1	0001
		2	0010
		8	1000
Group	2	5	0101
		6	0110
		9	1001
		10	1010
Group	3	7	0111
		14	1110

Comparison of adjacent groups

– Use X + X = X repeatedly between adjacent groups
– Those combined are checked off.
– Combine terms that have the same dashes and differ one in the number of 1's. (for column II and column III)

$$f = \underset{(1,\,5)}{a'c'd} + \underset{(5,\,7)}{a'bd} + \underset{(6,\,7)}{a'bc} + \underset{(0,\,1,\,8,\,9)}{b'c'} + \underset{(0,\,2,\,8,\,10)}{b'd'} + \underset{(2,\,6,\,10,\,14)}{cd'}$$

$$f = a'bd + b'c' + cd'$$

Determination of Prime Implicants

Column I

group 0	0	0000	✓
group 1	1	0001	✓
	2	0010	✓
	8	1000	✓
group 2	5	0101	✓
	6	0110	✓
	9	1001	✓
	10	1010	✓
group 3	7	0111	✓
	14	1110	✓

Column II

0, 1	000–	✓
0, 2	00–0	✓
0, 8	–000	✓
1, 5	0–01	
1, 9	–001	✓
2, 6	0–10	✓
2, 10	–010	✓
8, 9	100–	✓
8, 10	10–0	✓
5, 7	01–1	
6, 7	011–	
6, 14	–110	✓
10, 14	1–10	✓

Column III

0, 1, 8, 9	–00–
0, 2, 8, 10	–0–0
~~0, 8, 1, 9~~	~~–00–~~
~~0, 8, 2, 10~~	~~–0–0~~
2, 6, 10, 14	––10
~~2, 10, 6, 14~~	~~––10~~

Prime Implicants

- The terms that have not been checked off are called prime implicants.

  f = 0-01 + 01-1+011- + -00-

  + -0-0 + --10

  = $\underline{a'c'd}$ + a'bd + $\underline{a'bc}$ + b'c' +

  $\underline{b'd'}$ + cd'

- Each term has a minimum number of literals, but minimum SOP for f:

  f = a'bd + b'c' + cd'

  (a'bd, cd' => a'bc)

  (a'bd, b'c' => a'c'd)

  (b'c', cd' => b'd')

Definition of Implicant

- Definition
  - Given a function of F of n variables, a product term P is an implicant of F iff for every combination of values of the n variables for which P = 1, F is also equal to 1.
    - Every minterm of F is an implicant of F.
    - Any term formed by combining two or more minterms is an implicant.
    - If F is written in SOP form, every product term is an implicant.
- Example: $f(a,b,c) = a'b'c' + ab'c' + ab'c + abc = b'c' + ac$
    - If $a'b'c' = 1$, then $F = 1$, if $ac = 1$, then $F = 1$. $a'b'c'$ and ac are implicants.
    - If $bc = 1$, (but $a = 0$), $F = 0$, so bc is not an implicant of F.

Definition of Prime Implicant

– Definition
  - A prime implicant of a function F is a product term implicant which is no longer an implicant if any literal is deleted from it.
  - Example: f(a,b,c) = a'b'c' + ab'c' + ab'c + abc = b'c' + ac
    - Implicant a'b'c' is not a prime implicant. Why? If a' is deleted, b'c' is still an implicant of F.
    - b'c' and ac are prime implicants.
  - Each prime implicant of a function has a minimum number of literals that no more literals can be eliminated from it or by combining it with other terms.

Quine McClusky Procedure

- QM procedure:
  - Find all product term implicants of a function
  - Combine non-prime implicants.
    - Remaining terms are prime implicants.
  - A minimum SOP expression consists of a sum of some (not necessarily all) of the prime implicants of that function.
    - We need to select a minimum set of prime implicants.
  - If an SOP expression contains a term which is not a prime implicant, the SOP cannot be minimum.

Prime Implicant Chart

- Chart layout
  - Top row lists minterms of the function
  - All prime implicants are listed on the left side.
  - Place x into the chart according to the minterms that form the corresponding prime implicant.
- Essential prime implicant
  - If a minterm is covered only by one prime implicant, that prime implicant is called essential prime implicant. (9 & 14).
    - » Essential prime implicant must be included in the minimum sum of the function.

		0	1	2	5	6	7	8	9	10	14
(0, 1, 8, 9)	$b'c'$	X	X					X	X		
(0, 2, 8, 10)	$b'd'$	X		X				X		X	
(2, 6, 10, 14)	cd'			X		X				X	X
(1, 5)	$a'c'd$		X		X						
(5, 7)	$a'bd$				X		X				
(6, 7)	$a'bc$					X	X				

Selection of Prime Implicants

- √ Cross out the row of the selected essential prime implicants
- √ The columns which correspond to the minterms covered by the selected prime implicants are also crossed out.
- √ Select a prime implicant that covers the remaining columns. This prime implicant is not essential.

		0	1	2	5	6	7	8	9	10	14
(0, 1, 8, 9)	$b'c'$	X	X					X	X		
(0, 2, 8, 10)	$b'd'$	X		X				X		X	
(2, 6, 10, 14)	cd'			X		X				X	X
(1, 5)	$a'c'd$		X		X						
(5, 7)	$a'bd$				X		X				
(6, 7)	$a'bc$					X	X				

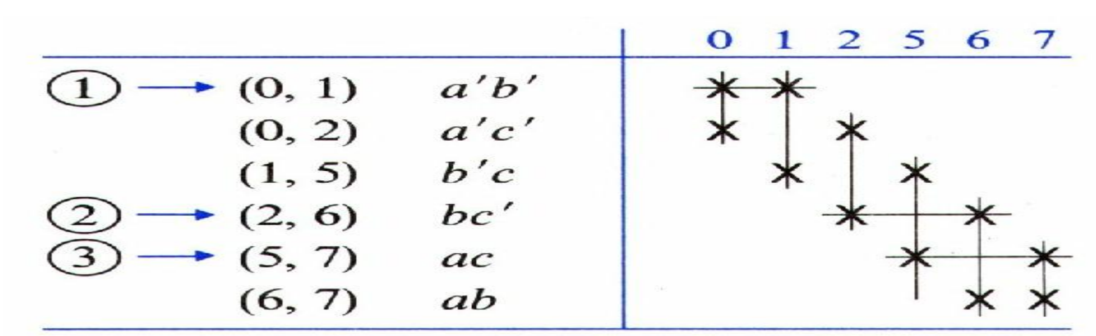
A Cyclic Prime Implicant Chart

- Two or more X's in every column.
- F = $\Sigma m(0,1,2,5,6,7)$
  - F = a'b' + bc' + ac. (by try and error). No guarantee for this to be minimum.

0	000
1	001
2	010
5	101
6	110
7	111

All checked off

0,1	00-
0,2	0-0
1,5	-01
2,6	-10
5,7	1-1
6,7	11-



Another Solution

- F = a'c' + b'c + ab
- Each minterm is covered by two different prime implicants.

			0	1	2	5	6	7
P_1	(0, 1)	$a'b'$	X	X				
P_2	(0, 2)	$a'c'$	X		X			
P_3	(1, 5)	$b'c$		X		X		
P_4	(2, 6)	bc'			X		X	
P_5	(5, 7)	ac				X		X
P_6	(6, 7)	ab					X	X

Petrick's Method

- A more systematic way to find all minimum solutions from a prime implicant chart.
- P is True when all the minterms in the chart have been covered. $P = f(P_1, P_2, \ldots)$
- Label each row with P_i.
  - P_i is true when the prime implicant in row P_i is included in the solution.
  - For column 0, we must choose either P_1 or P_2 in order to cover minterm 0. Thus $(P_1 + P_2)$ must be true.

			0	1	2	5	6	7
P_1	(0, 1)	$a'b'$	X	X				
P_2	(0, 2)	$a'c'$	X		X			
P_3	(1, 5)	$b'c$		X		X		
P_4	(2, 6)	bc'			X		X	
P_5	(5, 7)	ac				X		X
P_6	(6, 7)	ab					X	X

Petrick's Method (cont.)

- For column 0, we must choose either P_1 or P_2 in order to cover minterm 0. Thus $(P_1 + P_2)$ must be true.
- To cover minterm 1, $P_1 + P_3$ must be true, and etc.
- $P = (P_1+P_2)(P_1+P_3)(P_2+P_4)(P_3+P_5)(P_4+P_6)(P_5+P_6) = 1$
- This means: We must choose row P_1 or P_2, and row P_1 or P_3, and row P_2 or P_4, etc.

			0	1	2	5	6	7
P_1	(0, 1)	$a'b'$	X	X				
P_2	(0, 2)	$a'c'$	X		X			
P_3	(1, 5)	$b'c$		X		X		
P_4	(2, 6)	bc'			X		X	
P_5	(5, 7)	ac				X		X
P_6	(6, 7)	ab					X	X

Petrick's Method (cont.)

- Then we simplify

$P = (P_1+P_2)(P_1+P_3)(P_2+P_4)(P_3+P_5)(P_4+P_6)(P_5+P_6)$

$= (P_1 + P_2P_3)(P_4 + P_2P_6)((P_5 + P_3P_6);$

$(X+Y)(X+Z) = X + YZ)$

$= (P_1P_4 + P_1P_2P_6 + P_2P_3P_4 + P_2P_3P_6)(P_5 + P_3P_6)$

$= P_1P_4P_5 + P_1P_2P_5P_6 + P_2P_3P_4P_5 + P_1P_3P_4P_6 + P_2P_3P_6$

In other words, for P = 1 (to cover all minterms), we must choose row P_1 and P_4 and P_5 or row $P_1$and P_2 and P_5 and P_6 or etc ….

There are five to choose. We choose $P_1P_4P_5$ or $P_2P_3P_6$.

			0	1	2	5	6	7
P_1	(0, 1)	$a'b'$	×	×				
P_2	(0, 2)	$a'c'$	×		×			
P_3	(1, 5)	$b'c$		×		×		
P_4	(2, 6)	bc'			×		×	
P_5	(5, 7)	ac				×		×
P_6	(6, 7)	ab					×	×

Simplification of Incompletely Specified Functions

- An incompletely specified function

$$F(A, B, C, D) = \sum m(2, 3, 7, 9, 11, 13) + \sum d(1, 10, 15)$$

(the terms following d are don't cares)

The don't cares are treated like required minterms when finding the prime implicants:

1	0001 ✓	(1, 3)	00–1 ✓	(1, 3, 9, 11)	–0–1
2	0010 ✓	(1, 9)	–001 ✓	(2, 3, 10, 11)	–01–
3	0011 ✓	(2, 3)	001– ✓	(3, 7, 11, 15)	––11
9	1001 ✓	(2, 10)	–010 ✓	(9, 11, 13, 15)	1––1
10	1010 ✓	(3, 7)	0–11 ✓		
7	0111 ✓	(3, 11)	–011 ✓		
11	1011 ✓	(9, 11)	10–1 ✓		
13	1101 ✓	(9, 13)	1–01 ✓		
15	1111 ✓	(10, 11)	101– ✓		
		(7, 15)	–111 ✓		
		(11, 15)	1–11 ✓		
		(13, 15)	11–1 ✓		

The don't care columns are omitted when forming the prime implicant chart:

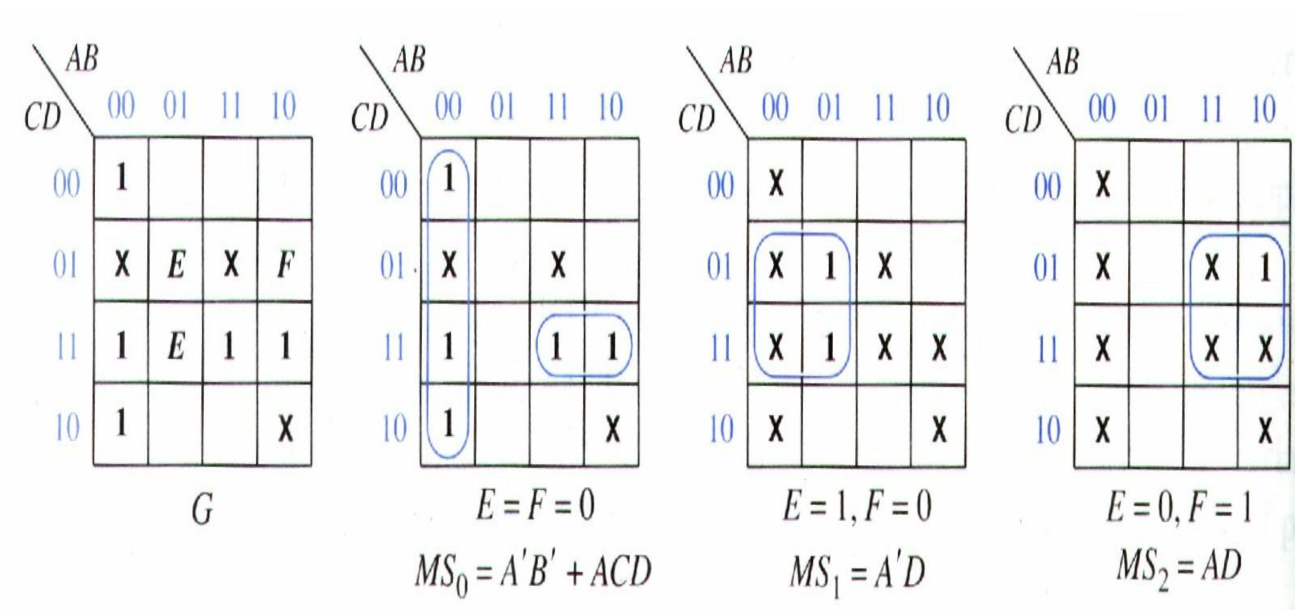
	2	3	7	9	11	13
(1, 3, 9, 11)		X		X	X	
*(2, 3, 10, 11)	X	X			X	
*(3, 7, 11, 15)		X	X		X	
*(9, 11, 13, 15)				X	X	X

* indicates an essential prime implicant.

$$F = B'C + CD + AD$$

Simplification Using Map-Entered Variables

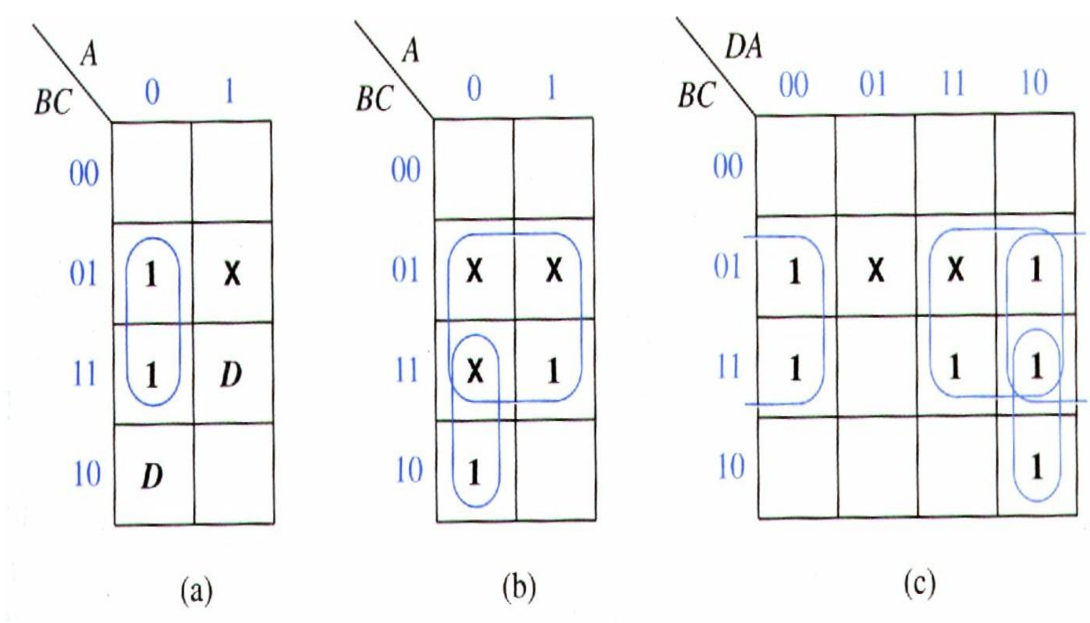
- Extend K-map for more variables.
  - When E appears in a square, if E = 1, then the corresponding minterm is present in the function G.
  - G (A,B,C,D,E,F) = $m_0 + m_2 + m_3 + Em_5 + Em_7 + Fm_9 + m_{11} + m_{15}$ + (don't care terms)



(a) (b) (c) (d)

Map-Entered Variable

- F(A,B,C,D) = A'B'C + A'BC + A'BC'D + ABCD + (AB'C), (don't care)
  - Choose D as a map-entered variable.
  - When D = 0, F = A'C (Fig. a)
  - When D = 1, F = C + A'B (Fig. b)
    - » two 1's are changed to x's since they are covered in Fig. a.
- F = A'C + D(C+A'B) = A'C + CD + A'BD



General View for Map-Entered Variable Method

- Given a map with variables P_1, P_2 etc, entered into some of the squares, the minimum SOP form of F is as follows:
- $F = MS_0 + P_1\, MS_1 + P_2 MS_2 + \ldots$ where
  - MS_0 is minimum sum obtained by setting $P_1 = P_2 .. =0$
  - MS_1 is minimum sum obtained by setting $P_1 = 1$, $P_j = 0$ $(j \neq 1)$, and replacing all 1's on the map with don't cares.

– Previously, G = A'B' + ACD + EA'D + FAD.